Thermodynamic limit of the first-order phase transition in the Kuramoto model

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In the Kuramoto model, a uniform distribution of the natural frequencies leads to a first-order (i.e., discontinuous) phase transition from incoherence to synchronization, at the critical coupling parameter K_c. We obtain the asymptotic dependence of the order parameter above criticality: $r - r_c \propto (K - K_c)^{2/3}$. For a finite population, we demonstrate that the population size N may be included into a self-consistency equation relating r and K in the synchronized state. We analyze the convergence to the thermodynamic limit of two alternative schemes to set the natural frequencies. Other frequency distributions different from the uniform one are also considered.

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I. INTRODUCTION

Synchronization is a universal phenomenon that plays an important role in all natural sciences as well as in technology [1,2]. In particular, the synchronization of populations of globally coupled oscillators with distributed natural frequencies has been an object of study since very early times, mainly in a biological context [3]. Later, it has found application in other areas, such as Josephson junctions [4], nanomechanics [5], etc. When increasing the coupling parameter, these systems undergo transitions from a totally incoherent state to a partially coherent state where part of the population becomes entrained sharing the same frequency. Interestingly, there are several analogies with the phase transitions in statistical mechanics [6]. Thus, one may define an order parameter that usually grows continuously from zero (the incoherent state) when the coupling parameter exceeds a threshold value, analogously to a second-order phase transition. Nonetheless, in some situations [7–9], mutual entrainment occurs in an abrupt way (a first-order phase transition). After an infinitesimal variation of the coupling strength a macroscopic (i.e., order N) part of the population becomes synchronized. One may speculate that first-order phase transitions may be of interest for practical applications, if one pursues a system exhibiting an abrupt off-on switch.

The most simple example of a first-order phase transition is found in the Kuramoto model [10] when the natural frequencies are uniformly distributed. In this case, it is known that all the population becomes synchronized in a single step [9,11]. We obtain here the asymptotic dependence of the order parameter after criticality, which exhibits a critical exponent 2/3.

Still, one of the open problems in the Kuramoto model is to fully understand the finite-N behavior. As we show below, the simplicity of the uniform frequency distribution allows to cope with finite-size effects in an original way, providing analytic and numerical results. Some of these results apply to other frequency distributions with compact support.

This paper is organized as follows. In Sec. II, we present the Kuramoto model, and show some numerical results that motivated this work. In Sec. III an infinite population is considered, finding the asymptotic dependence of the order parameter after criticality. In Sec. IV, we study finite ensembles, finding (a) an N-dependent formula for the order parameter, and (b) different convergence rates to the thermodynamic limit for alternative sampling schemes of the natural frequencies. Section V is devoted to analyzing frequency distributions different from the uniform one. Finally in Sec. VI the main conclusions of this work are summarized.

II. THE KURAMOTO MODEL

The Kuramoto model is probably the most studied model of synchronization in a population of oscillators with all-to-all coupling [12]. The state of each oscillator is described only by a phase variable (this stems from the the fact that, at small coupling, only the phase of a self-sustained oscillator is affected by the interaction). The phase θ_j of each oscillator satisfies the following ordinary differential equation:

$$\dot{\theta}_j = \omega_j + \frac{K}{N}\sum_{l=1}^{N} \sin(\theta_l - \theta_j) \tag{1}$$

where ω_j are the natural frequencies, and K is the parameter controlling the coupling strength.

To quantify the state of synchronization, Kuramoto proposed to use a complex-valued quantity (so-called order parameter to emphasize the relation with phase transitions)

$$re^{i\psi} = \frac{1}{N}\sum_{l=1}^{N} e^{i\theta_l}. \tag{2}$$

It allows us to set the governing equation (1) in the form

$$\dot{\theta}_j = \omega_j + Kr\sin(\psi - \theta_j). \tag{3}$$

If the natural frequencies are distributed (i.e., $\omega_j \neq \omega_{j'}$), synchronization only appears above some coupling threshold. Here, we consider the case of evenly spaced natural frequencies

$$\omega_j = -\gamma + \frac{\gamma}{N}(2j-1), \tag{4}$$

which, like all symmetric frequency distributions, can be assumed centered at zero (by going into a rotating frame if

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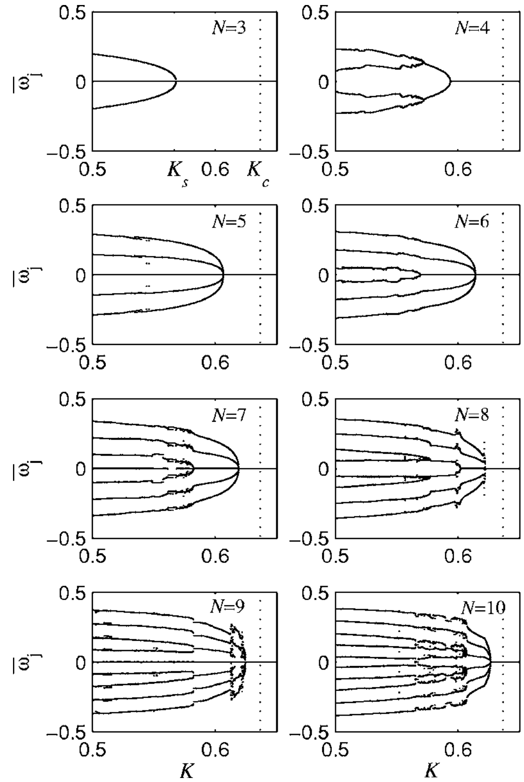

FIG. 1. Average frequencies ($\bar{\omega}_j$) as a function of the coupling for different population sizes. The natural frequencies are taken according to Eq. (4) with $\gamma=1/2$. By K_s we denote the value of K for the first splitting bifurcation. The critical point K_c in the thermodynamic limit ($N\rightarrow\infty$) is located at the dotted line.

necessary). Throughout this paper, the numerical integration of the Kuramoto model [Eq. (1)] is carried out by means of a fourth-order Runge-Kutta method with time step $\Delta t=0.1$.

Recently, Maistrenko *et al.* [13] studied the Kuramoto model with a small number of oscillators and natural frequencies distributed uniformly (or close to that). They found that the synchronized state robustly splits into several clusters with different average frequencies. It is shown there, and in Fig. 1, that for $N=3$ and 5, the synchronized state splits directly into N clusters. But, for values of N other than 3 and 5 the scenario is not so simple. As N increases, the number of splittings for going from one to N clusters, increases as well. Hereafter, we denote by K_s the coupling at the frequency splitting from the synchronized state. In congruence with the thermodynamic limit (see below), all the splittings [including the first one at $K=K_s(N)$] must accumulate at K_c as $N\rightarrow\infty$. K_c is the abrupt transition point for an infinite population.

III. INFINITE POPULATION

In this section, we briefly study the Kuramoto model for a uniform frequency distribution. Some of the formulas will be later compared to those obtained for a finite population in Sec. IV. We also go one step further, and deduce an explicit formula, with critical exponent 2/3, for the dependence of the order parameter after criticality.

In correspondence to the finite case Eq. (4), we consider a uniform density of the natural frequencies:

$$g(\omega)=\begin{cases}\dfrac{1}{2\gamma} & \text{for } |\omega|\leqslant\gamma,\\ 0 & \text{for } |\omega|>\gamma.\end{cases} \qquad (5)$$

We first note that due to the invariance under global rotation, for stationary solutions, we can set a vanishing phase for the order parameter in Eq. (2), $\psi=0$, without lack of generality. Kuramoto's classical analysis gives the order parameter equation

$$r=\langle e^{i\theta}\rangle=\langle\cos\theta\rangle\equiv\int_{-\infty}^{\infty}\cos\theta(\omega)g(\omega)d\omega. \qquad (6)$$

In the totally locked regime, we obtain then

$$r=\int_{-\gamma}^{\gamma}g(\omega)\sqrt{1-\frac{\omega^2}{K^2r^2}}d\omega \qquad (7)$$

$$\Rightarrow r=\frac{1}{2}\sqrt{1-\frac{\gamma^2}{K^2r^2}}+\frac{Kr}{2\gamma}\arcsin\left(\frac{\gamma}{Kr}\right). \qquad (8)$$

Equation (8) gives implicitly the dependence of r on K. A solution exists only for $Kr\geqslant\gamma$. At the critical point $K_cr_c=\gamma$, the locked solution disappears with $r_c=\pi/4$. The corresponding value of the coupling is $K_c=4\gamma/\pi$, which is precisely the value where the incoherent solution $r=0$ becomes unstable according to the classical result [14] for all unimodal distributions $K_c=2/\pi g(0)$, Two remarks are in order. First, in contrast with strictly unimodal distributions the transition is of first-order type: the order parameter "jumps" from zero to r_c. Second, at K_c *all* the population becomes entrained. This last remark is quite important because it simplifies both numerical and theoretical analyses. Also, note that when synchronized, the oscillator phases are spanned along an interval (of length π at $K=K_c$). When K is increased r grows from r_c to 1 in the $K\rightarrow\infty$ limit.

The first result of this paper is the dependence of r on K, just above the phase transition. First of all, we make a change of variables onto Eq. (6) as usual (see, e.g., [1,15]):

$$r=Kr\int_{\theta_{\min}}^{\theta_{\max}}\cos^2\theta\, g(Kr\sin\theta)d\theta, \qquad (9)$$

where $\theta_{\max}$ ($\theta_{\min}$) is the phase of the oscillator with frequency γ ($-\gamma$). After an expansion above criticality,

$$K=K_c+\delta K, \qquad (10)$$

$$r=r_c+\delta r, \qquad (11)$$

$$\theta_{\max}=-\theta_{\min}=\pi/2-\delta\theta, \qquad (12)$$

and discarding the trivial incoherent solution $r=0$, we get

$$1=\frac{K_c+\delta K}{2\gamma}\left(\frac{\pi}{2}-\delta\theta+\frac{1}{2}\sin(\pi-2\delta\theta)\right). \qquad (13)$$

An expansion of the sine function up to the cubic term yields

$$0=\frac{\pi}{2}\delta K-\frac{8\gamma}{3\pi}\delta\theta^3. \qquad (14)$$

Thus the problem reduces to finding $\delta\theta$, from Eq. (3);

$$\gamma=(K_c+\delta K)(r_c+\delta r)\sin(\pi/2-\delta\theta) \qquad (15)$$

$$\Rightarrow \delta\theta \approx \sqrt{\frac{8}{\pi}\delta r+\frac{\pi}{2\gamma}\delta K}. \qquad (16)$$

Introducing this expression into Eq. (14), a formula for the asymptotic dependence of δr on δK is obtained:

$$\delta r=\left(\frac{9\pi^7}{2^{17}\gamma^2}\right)^{1/3}\delta K^{2/3}-\frac{\pi^2}{16\gamma}\delta K. \qquad (17)$$

The result is compared in Fig. 2 to the exact solution, arising from numerically solving Eq. (8). It confirms that the order parameter grows from r_c with a power of $K-K_c$ with exponent $2/3$. To our knowledge, this exponent and expression (17) have not been reported before.

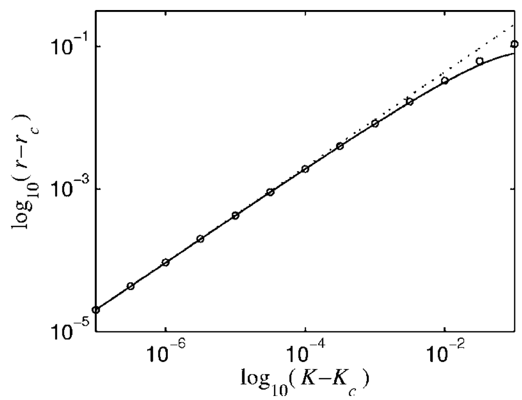

FIG. 2. Log-log dependence of r on K in a neighborhood of criticality for $\gamma=1/2$. The circles correspond to numerical solutions of the self-consistency condition (8), the solid line depicts Eq. (17), and the dotted straight line arises taking the leading $\delta K^{2/3}$ term only.

IV. FINITE POPULATION

Finite-size effects in the Kuramoto model have been previously considered in the literature. Among the different approaches, we may list the investigation of the divergence of fluctuations around criticality [6,16], the observation of ephemeral coherent structures in the incoherent state [17], and the reduction to a normal form in the case of identical natural frequencies with additive noise [18]. Also, very recently, Mirollo and Strogatz [19] have analyzed the (local) stability of the fully locked state for a finite population, finding that the locked solution is stable and disappears in a saddle-node bifurcation[1] (as already observed in [13] for small N).

In this section, we show that for the uniform frequency distribution finite-size effects—on the order parameter and on the loss of the synchronization—can be studied in a different way.

A. Dependence of the order parameter on K

For a finite population in the synchronized state ($K \geqslant K_s$), the order parameter is expressed [in correspondence with the integral form (7)] by

$$r=\frac{1}{N}\sum_{j=1}^{N}\sqrt{1-\frac{\omega_j^2}{K^2r^2}} \qquad (18)$$

where the order parameter r is a time-independent quantity. We devote the following lines to deducing an N-dependent self-consistency equation [that reduces to Eq. (8) in the $N\to\infty$ limit]. It is accurate provided that K is not too close to K_s.

According to Eq. (4), the natural frequencies of the finite population are taken with step $\Delta\omega=\omega_{j+1}-\omega_j=2\gamma/N$. Therefore, the order parameter (18) is equivalent to a Riemann sum with constant step of the integral found in the continuum limit (7). The discrepancy between both cases is, at the leading order, dependent on f'' [where $f(\omega)\equiv\sqrt{1-(\omega/Kr)^2}$]. Hence, for one Riemann box centered at ω_j,

$$\int_{\Delta\omega} f(\omega)d\omega=f(\omega_j)\Delta\omega+\frac{f''(\omega_j)}{24}\Delta\omega^3+O(\Delta\omega^5). \qquad (19)$$

Therefore, for the finite-N case we may approximate Eq. (18) by

$$r\approx\frac{1}{2\gamma}\int_{-\gamma}^{\gamma}\sqrt{1-\frac{\omega^2}{K^2r^2}}d\omega-\frac{\gamma^2}{6N^3}\sum_{j=1}^{N}f''(\omega_j). \qquad (20)$$

In this expression, the sum may be approximated by its corresponding integral,

$$\sum_{j=1}^{N}f''(\omega_j)=\frac{N}{2\gamma}\left(\int_{-\gamma}^{\gamma}f''(\omega)d\omega+O(N^{-2})\right)$$
$$\approx\frac{N}{2\gamma}f'(\omega)\Big|_{-\gamma}^{\gamma}=-\frac{N}{\gamma Kr\sqrt{K^2r^2/\gamma^2-1}}, \qquad (21)$$

and we obtain a self-consistency equation for a finite population of N oscillators:

$$r=\frac{1}{2}\sqrt{1-\frac{\gamma^2}{K^2r^2}}+\frac{Kr}{2\gamma}\arcsin\left(\frac{\gamma}{Kr}\right)+\frac{\gamma N^{-2}}{6Kr\sqrt{K^2r^2/\gamma^2-1}}. \qquad (22)$$

With respect to the equation for the thermodynamic limit [Eq. (8)], there is an additional term depending on N. Plots of the numerical solutions for $N=10$ and 100 are shown in Fig. 3. Equation (22) is deduced from the continuum equation, and therefore cannot intersect the line $r=\gamma/K$, because several terms explode. Note that adding more terms of the series in Eqs. (19) and (21) does not overcome this problem. This suggests that, unfortunately, the behavior very close to K_s cannot be deduced by simply manipulating the equations for an infinite population. Nonetheless, out of that region, the

[1] In the presence of an irrelevant degenerate eigenvalue at zero due to global phase-shift invariance.

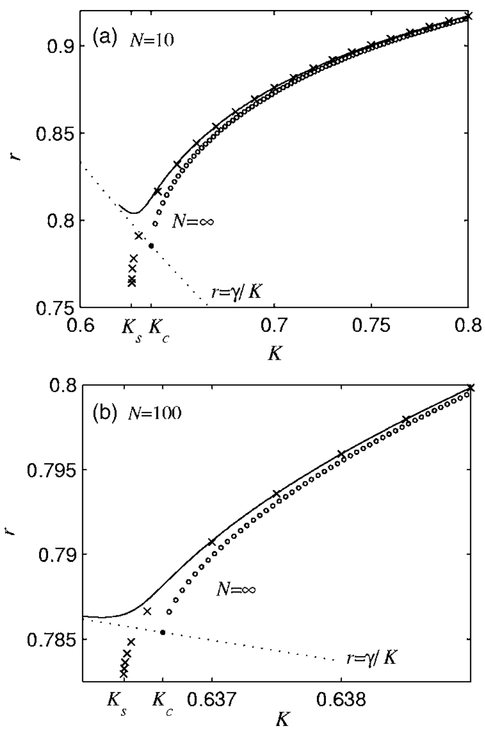

FIG. 3. Dependence of r on K for finite populations: $N=$ (a) 10 and (b) 100; $\gamma=1/2$ in both cases. Values obtained from a direct computation of the Kuramoto model are marked by $\times$. The first frequency splitting is observed at K_s (for $K<K_s$, r is no longer constant in time). The solid line is obtained solving Eq. (22) numerically. It matches the computed values, but fails when approaching the line $r=\gamma/K$. As a reference, the solution for an infinite population is shown with circles [after numerically solving Eq. (8)]; the critical point (K_c, r_c) is marked with a ● symbol.

solution of Eq. (22) reproduces the numerical results, even for such a relatively small number of oscillators as $N=10$ [Fig. 3(a)].

B. Thermodynamic limit of the first frequency splitting

The arrangement of the natural frequencies in Eq. (4) converges to the uniform frequency distribution (5). But, as explained above, this limit is nontrivial: a first-order phase transition is substituted, when N becomes finite, by a set of frequency-splitting bifurcations accumulating at K_c. Numerically, the study of these bifurcations is quite involved. Nonetheless, the point where, as K decreases, the first splitting occurs ($K=K_s$), can be accurately computed. This is possible because above K_s the system is in a fixed point state. Our simulations, Fig. 4, indicate that for the arrangement in Eq. (4) K_s converges to K_c^- according to a power law:

$$K_c - K_s(N) \propto N^{-\mu}. \qquad (23)$$

Note that K_s and K_c are both proportional to γ, so $\mu \approx 1.5$ is independent of γ.

The recipe followed to mimic the thermodynamic limit was to divide the frequency distribution $g(\omega)$ in N parts of equal area taking each ω_j at the center of each block.

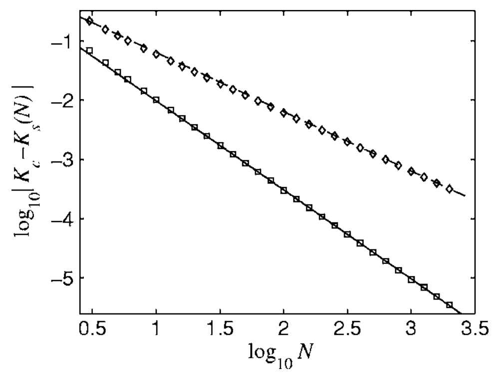

FIG. 4. Log-log dependence of the distance from K_s to K_c on the population size ($\gamma=1/2$). Squares and diamonds correspond to different arrangements of the natural frequencies, Eqs. (4) and (24), respectively. In the first case, the last two decades were fitted with a straight line, yielding a slope $-\mu=-1.502$. The dashed straight line $\log_{10}(4\gamma/\pi)-x$ arises from Eq. (25).

Nonetheless, there are many (infinite in fact) possible discrete arrangements with the same continuum limit, but for arrangement (4) μ is large enough to deduce the decay for other distributions, by just considering the variation of the effective γ. For instance, if the natural frequencies are taken as in [9,20]:

$$\omega_j = -\gamma + \frac{\gamma}{N-1} 2(j-1), \qquad (24)$$

one observes that K_s converges to K_c^+ with a power law (see the ◇'s in Fig. 4), but more slowly than for the arrangement in Eq. (4). As the extrema of Eq. (24) are fixed, a change in N varies the effective width of the equivalent continuous distribution: $\gamma_{eff}=\gamma+\gamma/(N-1)$. Hence, from Eq. (23) we get

$$K_s(N) = \frac{4\gamma_{eff}}{\pi} - \alpha N^{-\mu} \approx K_c + \frac{4\gamma}{\pi N} + O(N^{-\mu}), \qquad (25)$$

which agrees with the observed result (see the dashed line in Fig. 4).

V. OTHER FREQUENCY DISTRIBUTIONS

In this section, we briefly discuss the extension of the previous results to other frequency distributions supported on

TABLE I. Three frequency distributions considered in Sec. V.

	$g(\omega)$	K_s^∞
Parabolic	$\frac{3}{4\gamma^3}(\gamma^2-\omega^2)$	$\frac{32\gamma}{9\pi}$
Triangular	$\frac{\gamma-\|\omega\|}{\gamma^2}$	$\frac{6\gamma}{3\pi-4}$
Hat shaped	$\frac{2}{3\gamma}\left(\|\omega\| \leqslant \frac{\gamma}{2}\right)$ $\frac{1}{3\gamma}\left(\frac{\gamma}{2} < \|\omega\| \leqslant \gamma\right)$	$\frac{36\gamma}{8\pi+3\sqrt{3}}$

a finite interval $[-\gamma, \gamma]$. We first note that for nonuniform distributions the loss of complete synchronization and the critical point where the incoherent solution becomes unstable do not coincide: $K_s^{\infty} \equiv K_s(N \to \infty) > K_c$. In other words, there exists an intermediate range of partial entrainment in which one part of the population is synchronized whereas the remaining oscillators drift.

As model distributions we considered three unimodal distributions listed in Table I. As for the uniform distribution, the desynchronization point K_s^{∞} may be computed analytically using Eq. (7).

We focused on two simple sampling schemes to discretize distributions supported on a bounded interval: (i) $\int_{\tilde{\omega}_j}^{\tilde{\omega}_{j+1}} g(\omega) d\omega = 2\gamma/N, \tilde{\omega}_1 = -\gamma, \omega_j = (\tilde{\omega}_j + \tilde{\omega}_{j+1})/2$; (ii) $\int_{\omega_j}^{\omega_{j+1}} g(\omega) d\omega = 2\gamma/(N-1), \omega_N = -\omega_1 = \gamma$. Applying (i) and (ii) to a uniform distribution one gets the arrangements in Eqs. (4) and (24), respectively.

For both schemes and the three frequency distributions in Table I, the approach of the first frequency splitting to the thermodynamic limit satisfies a power law $|K_s^{\infty} - K_s(N)| \propto N^{-\mu}$, as occurred for the uniform distribution ($K_s^{\infty} = K_c$ in this case). For sampling (ii) we find that the value of the exponent is always $\mu \approx 1$. However for (i), different exponents arise, in contrast to $\mu \approx 1.5$ obtained for the uniform distribution: $\mu \approx 0.5$ for triangular and parabolic distributions, and $\mu \approx 1$ for the hat-shaped one. We have checked that this exponent arises for other distributions with an abrupt boundary $[g(\pm\gamma) > 0]$.

Another interesting power law is the shift of the order parameter in the synchronized state: $|r(K,N) - r(K,N=\infty)| \sim N^{-\nu}$. For sampling scheme (i) theoretical results may be obtained, using again arguments based on the Riemann sum. If $g(\pm\gamma) > 0$—e.g., uniform [see Eq. (22)] or hat-shaped distributions—one may obtain a formal solution[2] that yields $\nu = 2$. However for distributions that approach zero at $\omega = \pm\gamma$, the "Riemann-sum approach" is not valid due to divergences at $\omega = \pm\gamma$. One must, therefore, analyze these points separately. In particular, one obtains $\nu = 3/2$ for linearly decaying distributions $[g(\omega \to \pm\gamma) \sim (\gamma \mp \omega)$, e.g., parabolic and triangular distributions]. Also, our simulations indicate that for sampling (ii) $\nu \approx 1$, irrespective of the frequency distribution [for the uniform distribution $\nu = 1$ is straightforward due to the $O(N^{-1})$ effective shift of γ].

Finally, we note that a recipe similar to (i) consisting in taking the frequencies at the median (instead of the center) of each block[3] exhibits the same exponents μ, ν than scheme (i).

[2] $r = \int_{-\gamma}^{\gamma} g f \, d\omega - (N^{-2}/24) \int_{-\gamma}^{\gamma} [(g f'' + 2 g' f')/g^2] d\omega$.

[3] $\int_{\omega_j}^{\omega_{j+1}} g(\omega) d\omega = 2\gamma/N = 2 \int_{-\gamma}^{\omega_1} g(\omega) d\omega$.

VI. CONCLUSIONS

In the present paper, the first-order phase transition arising when imposing a uniform frequency distribution on the Kuramoto model has been studied. In the case of an infinite population, we have found an explicit asymptotic dependence of the order parameter after criticality, Eq. (17).

For a *finite* population, our first conclusion is that in contrast to strictly unimodal distributions (Cauchy, Gaussian, parabolic, etc.) of the natural frequencies, which exhibit transitions of second-order type, the thermodynamic limit is nontrivial for a uniform distribution. In the finite-N case, the synchronized state does not split directly into N clusters, but through a cascade of frequency splittings. To be congruent with the first-order phase transition predicted in the thermodynamic limit all the splittings must accumulate at K_c as $N \to \infty$.

The dependence of the order parameter r on the coupling K, has been expressed in an easy-to-compute formula, Eq. (22). In this formula, the population size N enters explicitly, and it allows, except very close to the desynchronization point, an accurate computation of the order parameter, even for small N.

Two sampling schemes to set the natural frequencies have been compared. The scheme we propose in Eq. (4) converges to the thermodynamic limit faster than another used in the literature [Eq. (24)]. The comparison is based on the different exponent of the power-law convergence for the point of the first frequency splitting ($\mu \approx 1.5$ vs $\mu = 1$) and the shift of the order parameter ($\nu = 2$ vs $\nu = 1$). Other frequency distributions (with compact support) different from the uniform one have been discussed. Among the infinite possible sampling schemes, those studied here appear the most natural ones to us. Nonetheless, further investigation is needed to assess the existence of an optimal sampling scheme to mimic the thermodynamic limit with a finite population.

In spite of the lack of structural stability under perturbations of the uniform frequency distribution (in the thermodynamic limit, but not in the finite case as proved in [13]), the results in this paper could be useful in order to understand more complicated schemes, like oscillator networks [21]. The use of a frequency distribution with an abrupt transition seems more suited to better resolve critical points in this kind of systems.

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